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B.E.M. WITH GALERKIN TECHNIQUE, HYPERSINGULAR AND VECTORIAL FORMULATION, FOR 2D POTENTIAL PROBLEMS TO CALCULATE THE TANGENTIAL DERIVATIVE USING LINEAR ELEMENTS.

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Abstract. *This work presents a computational methodology for Boundary Element Methods (B.E.M.) using a vectorial and hypersingular formulation to 2D potential problems (heat transfer, fluid transfer, sound, percolation etc.) applying the Galerkin technique and linear elements to calculate the potential, the normal derivative and the tangential derivative. The hypersingular formulation can furnish the tangential derivative directly, which can be useful when one want both the normal and tangential derivatives in order to obtain the resultant flux. Although had been subject of intensive studies the hypersingular formulation until has vary conceptual aspects not totally explored and a methodology not experimented. Here is studied the use of Galerkin technique and a vectorial formulation showing that the increase of work with double integration is compensated by less singularities in the first integration and none in the second.*

Key Words*:* *Boundary elements method, Hypersingular formulation, Galerkin integrals.*

1.INTRODUCTION.

B.E.M. is a computational method for several engineering problems and have developed largely the last years. The use of the hypersingular formulation is more recent and was used by Bézine 1976 and Stern 1978 to study plate bending. More recently it has been used as a alternative for the classic formulation or as a combination of both formulations (Ingler and Rudolph 1990, Telles and Prado 1993) or as an alternative for sub-regions in mechanical fracture analysis (Telles and Guimarães 1994) . Those works used the point collocation technique. The use of Galerkin method in hypersingular formulations is even more recent mostly due to L. J. Gray (1996 and others in press), for specific problems combining classical and hypersingular Galerkin formulations together, covering each one part of the boundary which was called symmetric Galerkin. Fleury and others (1998) presented a formulation and results for the Galerkin method applied to hypersingular BEM 2D using constant and discontinuous linear elements to obtain the potential and the normal derivative. The basic formulations used were taken from the collocation method for a vectorial hypersingular formulation developed by Mansur and others (1997). The Galerkin method needs a double integration. Singularities problems in the first integration are less than in the collocation method as the source points are never in the corners. In the second integration there are no singularities anymore.

Some applications are presented, with results and compared with others obtained with classic and hypersingular formulations using collocation point. Results are discussed and analyzed in other to choose better methods in each case and in order to clarify some conception aspects.

2. METHOD

The method is very similar to the collocation point method and the classical formulation of Galerkin method. The difference to the collocation point is that there is another integral do be made over the source element. We can follow the usual steps as the collocation point but having in mind that there is no single source point and that the second integral will create double coefficients for each single point. By the other hand the point in the extreme of each source element do not represent a source of singularities to integrate the just next element field.

Below is the weighted hypersingular vectorial formulation for the normal derivative:

$$\int_{\Gamma s} W\Big[\ Cf(p_n, p_t) + VP\int_{\Gamma x} p^*_{n,n}(\underline{s},\underline{x})\big[u(\underline{x}) - u(\underline{s})\big]d\Gamma(\underline{x}) - \int_{\Gamma x} u^*_{,n}(\underline{s},\underline{x})\, p_n(\underline{x})\, d\Gamma(\underline{x})\ \Big]d\Gamma(\underline{s}) = 0 \tag{1}$$

And for the tangential derivative has the below formulation:

$$\int_{\Gamma s} W\Big[\ Cf(p_n, p_t) + \int_{\Gamma x} p^*_{n,t}(\underline{s},\underline{x})\big[u(\underline{x}) - u(\underline{s})\big]d\Gamma(\underline{x}) - VP\int_{\Gamma x} u^*_{,t}(\underline{s},\underline{x})\, p_n(\underline{x})\, d\Gamma(\underline{x})\ \Big]d\Gamma(\underline{s}) = 0 \tag{2}$$

Where: W is the weight functions , Cf(pn, pt) represents the free terms, $\underline{s}$ is the source point ordinates, and $\underline{x}$ the field ones, r is the distance between $\underline{s}$ and $\underline{x}$, $\underline{v}$ is a unity vector pointed from $\underline{s}$ to $\underline{x}$, $\underline{n}(\underline{x})$ is the unity normal vector in $\underline{x}$, $\underline{n}(\underline{s})$ is the unity normal vector in $\underline{s}$, $u(\underline{x})$ is the potential in $\underline{x}$, $\underline{u}(\underline{s})$ is the same in $\underline{s}$, $p_n(\underline{x})$ is the normal derivative of $u(\underline{x})$, $p_t(\underline{x})$ is the tangential derivative of $u(\underline{x})$, Γ_x denotes the boundary, Γ_s is the sourse element. VP indicates that this integral can be singular. And:

$$p^*_{n,n}(\underline{s},\underline{x}) = \frac{1}{2\pi r^2}\left\{2\big(\underline{v}\bullet\underline{n}(\underline{x})\big)\big(\underline{v}\bullet\underline{n}(\underline{s})\big) - \underline{n}(\underline{x})\bullet\underline{n}(\underline{s})\right\} \qquad u^*_{,n}(\underline{s},\underline{x}) = \frac{1}{2\pi r}\big(\underline{v}\bullet\underline{n}(\underline{s})\big) \tag{1A}$$

$$p^*_{n,t}(\underline{s},\underline{x}) = \frac{1}{2\pi r^2}\left\{2\big(\underline{v}\bullet\underline{n}(\underline{x})\big)\big(\underline{v}\bullet\underline{t}(\underline{s})\big) - \underline{n}(\underline{x})\bullet\underline{t}(\underline{s})\right\} \qquad u^*_{,t}(\underline{s},\underline{x}) = \frac{1}{2\pi r}\big(\underline{v}\bullet\underline{t}(\underline{s})\big) \tag{2A}$$

The Γ_x and Γ_s coordinates are changed to η and η' local coordinates so $d\Gamma_x = lj/2\ d\eta$ and $d\Gamma_s = li/2\ d\eta'$. Classically we have two functional points for each element, two interpolation functions $\phi1(\eta)$ e $\phi2(\eta)$ related to them and two weight functions $\phi1(\eta')$ e $\phi2(\eta')$ related with the points of the source element. The interpolation and weight functions are :

$$\phi1(\eta)=\frac{1-\eta}{2} \qquad \phi2(\eta)=\frac{1+\eta}{2} \qquad \text{and} \qquad \phi1(\eta')=\frac{1-\eta'}{2} \quad \phi2(\eta')=\frac{1+\eta'}{2} \tag{3}$$

The interpolated variables take the form:

$$\begin{aligned}
&u(x)=\phi1(\eta)u1j+\phi2(\eta)u2j\\
&u(s)=\phi1(\eta')u1i+\phi2(\eta')u2i\\
&p_n(x)=\phi1(\eta)p_n1j+\phi2(\eta)p_n2j\\
&p_n(s)=\phi1(\eta')p_n1i+\phi2(\eta')p_n2i_n\\
&p_t(s)=\phi1(\eta')p_t1i+\phi2(\eta')p_t2i
\end{aligned} \tag{4}$$

Where j denotes the field, i the source, 1 to the first point element, 2 to the second, η the local ordinate, η' the source local element ordinate. We can consider all points inside the elements when calculating, because we are going to use Gauss points and the extreme points are, by this way, not relevant. So the free term becomes $0{,}5p_n(\underline{s})$ for the normal formulation and $0{,}5p_t(\underline{s})$ for the tangential one:

$$\int_{-1}^{+1}\left|\phi1(\eta')\,\phi2(\eta')\right|^T\frac{li}{2}\left[0{,}5p_n(s)+\int_{-1}^{+1}p^*_{n,n}(s,x)[u(x)-u(s)]\frac{lj}{2}d\eta-\int_{-1}^{+1}u^*_{,n}(s,x)p_n(x)\frac{lj}{2}d\eta\right]d\eta'=0 \tag{5}$$

$$\int_{-1}^{+1}\left|\phi1(\eta')\,\phi2(\eta')\right|^T\frac{li}{2}\left[0{,}5p_t(s)+\int_{-1}^{+1}p^*_{n,t}(s,x)[u(x)-u(s)]\frac{lj}{2}d\eta-\int_{-1}^{+1}u^*_{,t}(s,x)p_n(x)\frac{lj}{2}d\eta\right]d\eta'=0 \tag{5A}$$

li/2 is constant for each source element and can be eliminated.

Integrals can been made separately and give the below coefficients:

2.1 - For the free terms (using p for p_n or p_t) and indicating 1 for $\phi1$ and 2 for $\phi2$:

$$B(1,1)p(s)1i = p\ (s)1i\ 0{,}5\int_{-1}^{+1}\phi1(\eta')^2d\eta' \qquad B(1,2)p(s)2i = p(s)2i\ 0{,}5\int_{-1}^{+1}\phi1(\eta')\phi2(\eta')d\eta'$$

$$B(2,1)p\ (s)1i = p(s)1i\ 0{,}5\int_{-1}^{+1}\phi2(\eta')\phi1(\eta')d\eta' \qquad B(2,2)p(s)2i = p(s)2i\ 0{,}5\int_{-1}^{+1}\phi2(\eta')^2d\eta' \tag{6}$$

which integrated are:

$$B(1,1)=\frac{1}{3} \qquad B(1,2)=\frac{1}{6} \quad \text{and} \qquad B(2,1)=B(1,2) \quad e \quad B(2,2)=B(1,1) \tag{7}$$

There are two coefficients for p(s)1i and two for p(s)2i

2.2 - For the coefficients that multiplies p_n we have, for the no singular integrals:

Developing equation (5) or (5A) we can see that we have more than one coefficient for each point and we have to add them, as explained below:
From equation (5) the AGs, coefficients of p_n , are calculate by the Gauss method. As an example when the weighting function is ϕ1(η') we have:

$$AG(p_n 1j, p_n 2j) = \int_{-1}^{+1} \phi 1(\eta') \left[\int_{-1}^{+1} u_{,n}^{*}(s,x) [\phi 1(\eta) p_n 1j(x) + \phi 2(\eta) p_n 2j(x)] \frac{lj}{2} d\eta \right] d\eta' \tag{8}$$

The integrals can be made by parts indicating the results by coefficients using as indices the number 1 when was used ϕ1(η) and 2 when was used ϕ2(η). As a example one can do the first integration using some point located in the field element denoted by k :

$$AG1\; p_n 1j = p_n 1j \int_{-1}^{+1} \frac{1}{2\pi r} (\underline{v} \bullet \underline{n}(\underline{x}))_k \frac{lj}{2} \phi 1(\eta) d\eta \tag{9}$$

The second integration is made, also using Gauss method and using 1and 2 for ϕ1(η') and ϕ2(η'):

$$AG(1,1)\, p_n 1i(x) = p_n 1i(x) \sum_{k=1}^{NE} \phi 1(k) * AG1(k) * OME(k) \tag{10}$$

and so on for the others coefficients AG(1,2),AG(2,1) and AG(2,2).

Here we notice that there is four coefficients for each derivative instead of two when collocation point is used.

Be a element number 4 which has point 4 at the left and point 5 at the right side. Taking point 5 of element 4, as an example, we can see that it will have four coefficients. Two coefficients when the integration of its interpolation function at the left side will be integrate again, once with ϕ1(η') and also with ϕ2(η'). The same will happen with the integration with its interpolation function at the right side (in element 5). Because of this double integration we will have four coefficients that have to be added as they refer to the same derivative at point 5. By the other hand if point 4 of element 4 is in a corner it will be treated as a double point and only have a interpolation function and its coefficient will be compound only by two terms.

After those summation we can indicate the coefficients by G(1,1), G(1,2), G(1,3) … for the ones obtained for the first matrix line (when the element 4 is the source) and so on.

This operation has to be done for all type of coefficient as AF, AH, AE, AC and AD which after the summations became F, H, E, C and D .

From equation (5A), following the same sequence, for the no singular terms we have:

$$AF(p_n 1j, p_n 2j) = \int_{-1}^{+1} \phi 1(\eta') \left[\int_{-1}^{+1} u_{,t}^{*}(s,x) [\phi 1(\eta) p_n 1j(x) + \phi 2(\eta) p_n 2j(x)] \frac{lj}{2} d\eta \right] d\eta'$$

(8A)

$$AF1C = \int_{-1}^{+1} \frac{1}{2\pi r}\left(\underline{v} \bullet \underline{t}(\underline{x})\right)\frac{lj}{2}\phi 1(\eta)d\eta \qquad (9A)$$

$$AF(1,1)\,p_n 1i(x) = p_n 1i(x)\sum_{k=1}^{NE} \phi 1(k) * F1C(k) * OME(k) \qquad (10A)$$

2.3 - For the terms that multiplies u(x) and -u(s)

The same operation is done for the coefficient that multiplies u(x) and are not singular. As example using the weighting function ϕ1(η'):

$$u1j(\underline{x}) * AH(1,1) = u1j(\underline{x})\int_{-1}^{+1}\phi 1(\eta')[\int_{-1}^{+1} p^*_{n,n}(s,x)\frac{lj}{2}\phi 1(\eta)]\,d\eta' \qquad (11)$$

$$u1j(\underline{x}) * AE(1,1) = u1j(\underline{x})\int_{-1}^{+1}\phi 1(\eta')[\int_{-1}^{+1} p^*_{n,t}(s,x)\frac{lj}{2}\phi 1(\eta)]\,d\eta' \qquad (11A)$$

and so on for the others, AH(1,2), AH(2,1),AH(2,2) and AE(1,2),AE(2,1),AE(2,2).

The coefficient that multiplies the potential value of the first source point -u1i(s) is :

$$AC1C\ u1i(\underline{s}) = u1i(\underline{s})\int_{-1}^{+1} p^*_{n,n}(s,x)\frac{lj}{2}\phi 1(\eta)d\eta \qquad (12)$$

$$AD1C\ u1i(\underline{s}) = u1i(\underline{s})\int_{-1}^{+1} p^*_{n,t}(s,x)\frac{lj}{2}\phi 1(\eta)d\eta \qquad (12A)$$

and for the second u2i(s) :

$$AC2C\ u2i(\underline{s}) = \int_{-1}^{+1} p^*_{n,n}(s,x)\frac{lj}{2}\phi 2(\eta)d\eta \qquad (13)$$

$$AD2C\ u2i(\underline{s}) = \int_{-1}^{+1} p^*_{n,t}(s,x)\frac{lj}{2}\phi 2(\eta)d\eta \qquad (13A)$$

Integrated by the Gauss method

$$AC(1,1) = \sum_{k=1}^{NE} C1C(k) * \phi 1(k) * OME(k) \qquad (14)$$

$$AD(1,1) = \sum_{k=1}^{NE} D1C(k) * \phi 1(k) * OME(k) \qquad (14A)$$

and so on for the others. Observe that those coefficients appears in all elements integrals because of the [u(x)-u(s)] term. All coefficient ACs related to the same source have to be added together and then subtract from the terms of the principal diagonal of the H matrix (H(i,i)) whose terms are related to the same derivative value. The same is done for all coefficients ADs and them subtract from the F matrix (F(i,i)). There is another way to do this, that is to make the subtraction of ACs and ADs from AHs and Afs before they were added and became Hs and Fs.

2.4 - Singular integrals:

The singular integrals have to be made analytically.
When the source element is the same as the field element (j = i) we can have singularities, but as $\underline{v} \bullet \underline{n}(\underline{s}) = 0$, $\underline{n}(\underline{s}) \bullet n(\underline{s}) = 1$, $\underline{v} \bullet \underline{t}(\underline{s}) = 1$ *and* $\underline{n}(\underline{s}) \bullet \underline{t}(\underline{s}) = 0$, then:

$$p^*_{n,n}(\underline{s},\underline{s}) = \frac{1}{2\pi r^2} \quad e \quad u,^*_n(\underline{s},\underline{s}) = 0 \tag{15}$$

$$p^*_{n,t}(\underline{s},\underline{s}) = 0 \quad e \quad u,^*_n(\underline{s},\underline{s}) = \frac{1}{2\pi r} \tag{15A}$$

Thus G(i,i), coefficients of $p_n(\underline{s})$, are null. They are substituted by the coefficients produced by the free terms (B type), that also multiplies $p_n(\underline{s})$, changing the signal.
The E(i,i), coefficients of $u(\underline{s})$, are also null.
Coefficient H when using as weighting function $\phi 1(\eta')$ is:

$$AH(1,1) = \int_{-1}^{+1} \phi(1)\eta') \left[\int_{-1}^{+1} \frac{1}{2\pi r^2} [u(x) - u(s)] \frac{lj}{2} d\eta \right] d\eta' \tag{16}$$

We substitute in equation (16) those values from equation (4) and make simplifications. To do the first integration we divide the element in two parts: first from -1 to η', and after from η' to +1. Thus the result will be a function of η'. The distance r between $\underline{s}$ and $\underline{x}$ in the first region is r=(η'–η)li/2 and in the second region is r= (η–η')li/2. Singular parts are eliminated using finite part or Cauchy principal value. One have:

$$AH1C = \frac{1}{2\pi li} \left[\int_{-1}^{\eta'} \frac{1}{(\eta' - \eta)} d\eta - \int_{\eta'}^{+1} \frac{1}{(\eta - \eta')} d\eta \right] \tag{17}$$

$$AH1C = \frac{1}{2\pi li} [\ln(\eta'+1) - \ln(1 - \eta')] \tag{18}$$

Repeating the same operation for H2C :

$$AH2C = \frac{1}{2\pi li} [-\ln(\eta'+1) + \ln(1 - \eta')] \tag{19}$$

The second integration has not singularities and can be integrate by Gauss method using η' as k a Gauss point. Then for this coefficient we have :

$$AH(1,1) = \sum_{k=1}^{NG} AH1C(k) * \phi 1(k) * OME(k) \tag{20}$$

$$AH(1,2) = \sum_{k=1}^{NG} AH2C(k) * \phi 1(k) * OME(k)$$

$$AH(2,1) = \sum_{k=1}^{NG} AH1C(k) * \phi 2(k) * OME(k)$$

$$AH(2,2)=\sum_{k=1}^{NG} AH2C(k)*\phi 2(k)*OME(k)$$

Observe that the subtraction -u(s) was included in this integration.

Repeating the same operations to find the E(i,i) singular coefficient we find for:

$$AE(1,1)=\int_{-1}^{+1}\phi(1)\eta')\left[\int_{-1}^{+1}\frac{1}{2\pi}\frac{(\underline{v}\bullet\underline{t})}{r}\frac{(1-\eta)}{2}\frac{lj}{2}d\eta\right]d\eta' \qquad (16A)$$

which gives:

$$AE1C=\frac{1}{4\pi}\left[(\eta'-1)\ln(\eta'+1)-(1+\eta')\ln(1-\eta')-2\right] \qquad (18A)$$

$$AE2C=\frac{1}{4\pi}\left[(1-\eta')\ln(\eta'+1)+(1+\eta')\ln(1-\eta')+2\right] \qquad (19A)$$

From those equations by a Gauss method integration as shown in equations (20) we can have AE(1,1), AE(1,2), AE(2,1) and AE(2,2) .

Coefficients H and G are used to find all normal derivatives and all potential values. Tangential derivative are calculated with a post - processing program using E and F coefficients.

3. EXAMPLE

We show two applications.

The first example is the same problem showed by Brebbia e Dominguez (1989). A flux of heat flow trough a squared boundary with a side of 6 units. There is a temperature of 300 units in the left side and a temperature of zero units in the opposite side. Trough the laterals there is no flux. The analytic solution is u(x)= 300(1-x/6) trough the flux and $\partial u/\partial x$ is constant 50 units including $\partial u/\partial n$ in the borders, being negative in the entrance and positive in the exit. $\partial u/\partial n$ is zero in the laterals.

The second example is made by inserting a circle with 2 units of radius centered in the middle of the previous squared example, thus we have a curved boundary, different normal and tangential flux for each boundary point. The boundary will not have corners and neither the temperature neither the flux are constant in the boundary. The boundary temperature follows the equation u(x)= 300(1-x/6). This example is treat as a Dichlet problem.

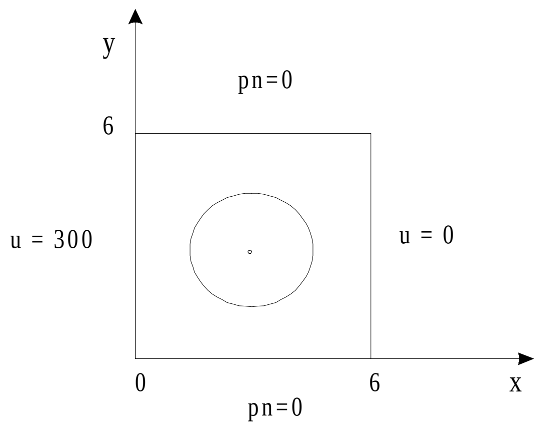

Figure 1 - Figure shows both examples sketch

Results are presented in graphics followed by comments. In all graphics the analytic result are presented by a full line.

First example. Square boundary

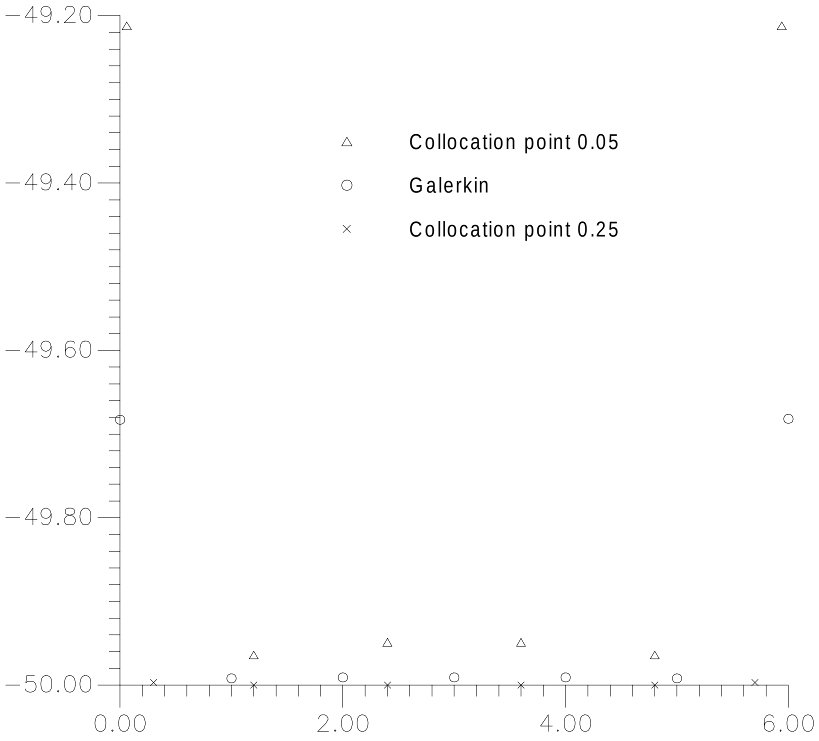

Figure 2 - This figure shows the normal derivative value in the square vertical face

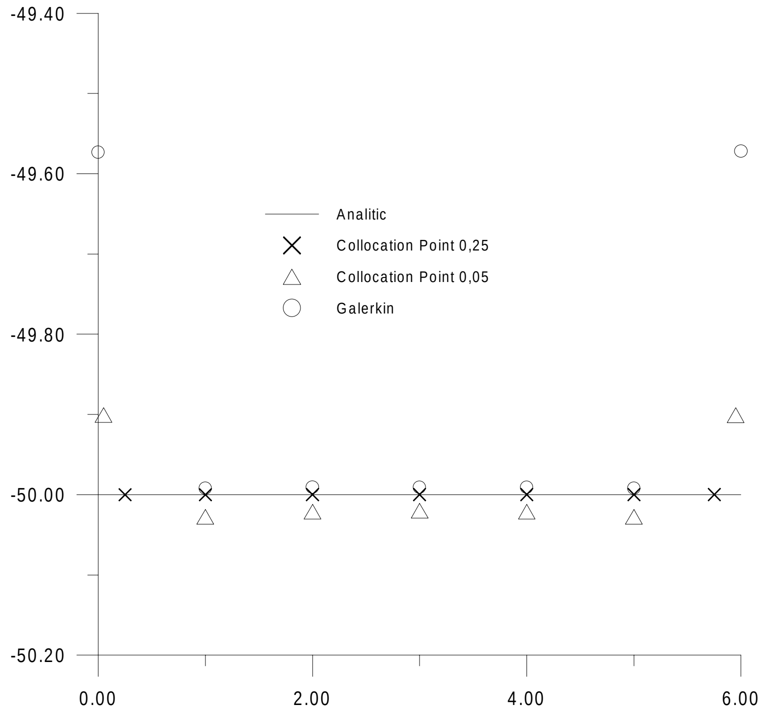

Figure 3 - This figure shows results of tangential derivative along the horizontal face.

In figure 2 scale was very enlarged to show little discrepancies. The circles represent points obtained with the Galerkin method made with 28 points, 8 Gauss points for the second integration, 12 Gauss points for the field integration. We can see that the presence of a corner causes a error of 0,6% in the point located just over the corner. The x represent values obtained with hypersingulat collocation point method, with the extremes points moved to the interior to avoid enormous error caused by the presence of a corner. The displacement, for collocation point method, was of 0.25 of the total element length. This method has very good result but one do not know the derivative just in the corner The triangles represents points of hypersingular collocation point method where de displacement of the corners points were of 0.05. This causes a great error in those two points and also create error in all other points.

Figure 3 is also enlarged. The same representation was used. Points not in the extremes result very close to the analytic In the corners, again, Galerkin hypersingular method has some error just over the corner point. The collocation point with corner points moved 0.25 inner to the element has better result but has no value for the corner point. The collocation point with corner points moved 0.05 has not so god results and the bad effects of the corner make the inner points also worst.

Second example. Circle boundary.

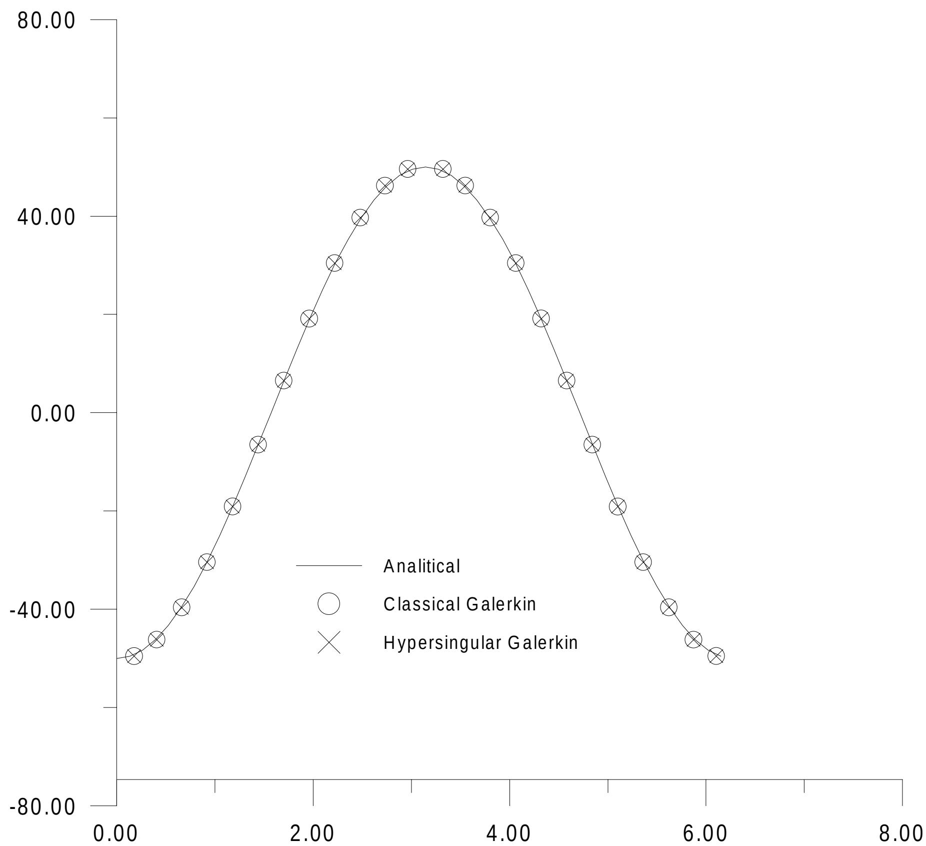

Figura 4 -This figure shows the result of normal derivative in the circle boundary. In vertical axis are the normal derivative values, in horizontal axis are the angle values in rad.

Figure 4 shows the analytical results in a full line, little circles represents results with a classical Galerkin formulation and little crosses represents results with hypersingular Galerkin formulation. Results, without the descontinuity caused by corners are exact, making difficult to see the crosses inside the circles.

Results for tangential derivative in the boundary circle are also good for both problems, as there are no corner discontinuities.

4.CONCLUSIONS

Results show that the Galerkin method is so good as the collocation point. In the corners as in other methods Galerkin method is not able to compensate the discontinuity of the gradient potential, but with better results. Do not treat the extremes points of the source elements as a singularity source for the contiguous elements makes no difference for the results, when the points are not to close to each other. Better results for the corner points for collocation point probably could be obtained using the special corner equations show by Mansur and others 1997 and for Galerkin method using Telles coordinates transformation 1993.

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